$$(2)\qquad \begin{aligned} s_1' &= as_1+bs_2+cs_3+ds_4,\\ s_2' &= -bs_1+as_2+ds_3-cs_4,\\ s_3' &= -cs_1-ds_2+as_3+bs_4,\\ s_4' &= -ds_1+cs_2-bs_3+as_4, \end{aligned}$$

where

$$a=x_1-x_2,\ b=x_3-x_4,\ c=x_5-x_6,\ d=x_7-x_8.$$

It follows as before that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, a, b, c, d$ are linearly independent functions of $x_1, x_2, \ldots, x_8$.

Let first the quantity $i\equiv\sqrt{-1}$ belong to the field F and set

$$\eta_5=s_1-is_2,\ \eta_6=s_3+is_4,\ \eta_7=-s_3+is_4,\ \eta_8=s_1+is_2,$$

so that $\eta_5, \eta_6, \eta_7, \eta_8$ are linearly independent functions of s_1, s_2, s_3, s_4. Then the transformation defined by (1) and (2) takes the canonical form

$$(3)\qquad \begin{cases} \eta_1'=\sigma_1\eta_1,\ \eta_3'=\sigma_3\eta_3,\ \eta_5'=\kappa\eta_5+\lambda\eta_6,\ \eta_7'=\kappa\eta_7+\lambda\eta_8,\\ \eta_2'=\sigma_2\eta_2,\ \eta_4'=\sigma_4\eta_4,\ \eta_6'=\mu\eta_5+\nu\eta_6,\ \eta_8'=\mu\eta_7+\nu\eta_8, \end{cases}$$

where

$$\kappa=a+ib,\ \lambda=c-id,\ \mu=-c-id,\ \nu=a-ib.$$

In particular, the determinant of transformation (3), and hence of matrix M, equals

$$\sigma_1\sigma_2\sigma_3\sigma_4(\kappa\nu-\lambda\mu)^2\equiv\sigma_1\sigma_2\sigma_3\sigma_4(a^2+b^2+c^2+d^2)^2.$$

As a corollary, we derive for the determinant of (2) the known formula

$$\begin{vmatrix} a & b & c & d\\ -b & a & d & -c\\ -c & -d & a & b\\ -d & c & -b & a \end{vmatrix}=(a^2+b^2+c^2+d^2)^2.$$

Suppose next that i does not belong to the field F. Then the transformation T of matrix M of coefficients belonging to F cannot be reduced to the canonical form (3) by means of a transformation of variables with coefficients in F. In fact, $a^2+b^2+c^2+d^2$ cannot be expressed in the form $\kappa\nu-\lambda\mu$, where $\kappa, \lambda, \mu, \nu$ are linear functions of a, b, c, d with coefficients in F (not containing i). As the canonical form of T for the field F, we therefore take the transformation defined by (1) and (2). For the enlarged field $F(i)$, obtained by adjoining i to F, we may take as the canonical form either (3) or, preferably, the transformation

$$(3')\qquad \begin{cases} \eta_1'=\sigma_1\eta_1,\ \eta_3'=\sigma_3\eta_3,\ \eta_5'=\kappa\eta_5+\lambda\eta_6,\ \eta_7'=\kappa\eta_7+\lambda\eta_8,\\ \eta_2'=\sigma_2\eta_2,\ \eta_4'=\sigma_4\eta_4,\ \eta_6'=-\bar{\lambda}\eta_5+\bar{\kappa}\eta_6,\ \eta_8'=-\bar{\lambda}\eta_7+\bar{\kappa}\eta_8, \end{cases}$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are arbitrary quantities in F, while κ, λ are arbitrary quantities in $F(i)$, $\bar{\kappa}$ denoting the conjugate of κ.

4. We may now readily determine the order of the group* of all transformations of matrix M of coefficients in the Galois Field of order p^n, $p>2$.

Let first i belong to the $GF[p^n]$, so that p^n-1 is a multiple of 4, the period of i. We have then to determine the number of the transformations (3) of determinant not zero. Here $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \kappa, \lambda, \mu, \nu$ are any elements of the $GF[p^n]$ such that

$$(4) \qquad \sigma_1\sigma_2\sigma_3\sigma_4(\kappa\nu-\lambda\mu)^2 \neq 0,$$

so that the order of the group is $(p^n-1)^4.(p^{2n}-1)(p^{2n}-p^n)$.

Let next i be not in the $GF[p^n]$, so that it serves to extend the $GF[p^n]$ to the $GF[p^{2n}]$. We have then to determine the number of transformations (3′) of the determinant not zero. Here $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are any elements $\neq 0$ of the $GF[p^n]$, while κ and λ are any elements of the $GF[p^n]$ such that

$$\triangle \equiv \kappa\bar{\kappa} + \lambda\bar{\lambda} \equiv \kappa^{p^n+1} + \lambda^{p^n+1} \neq 0.$$

But $\triangle = 0$ gives $\kappa = \tau\lambda$, where $\tau^{p^n+1} = -1$. Hence there are

$$1+(p^{2n}-1)(p^n+1)$$

sets κ, λ making $\triangle = 0$. Subtracting this sum from p^{4n}, the total number of sets κ, λ in the $GF[p^{2n}]$, we obtain $(p^{2n}-1)(p^{2n}-p^n)$ as the number of sets κ, λ making $\triangle \neq 0$. Hence the order is again†

$$(p^n-1)^4.(p^{2n}-1)(p^{2n}-p^n).$$

5. Suppose finally that the field F has modulus 2 The four functions in §3 which T multiplies by constants are now identical. We seek eight linear functions $\eta_1, \ldots, \eta_8$ which are linearly independent functions modulo 2 of

*In view of the canonical form (3) or (3′), the totality of the transformations T evidently possesses the group property.

†Another method, not assuming the independence of $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \kappa, \lambda, \mu, \nu$, consists in finding the number of sets $x_1, \ldots, x_8$ for which

$\sigma_1 \neq 0,\ \sigma_2 \neq 0,\ \sigma_3 \neq 0,\ \sigma_4 \neq 0,\ \kappa\nu-\lambda\mu = 0$	$p^n(p^{2n}+p^n-1)(p^n-1)^4$
$\sigma_1 \neq 0,\ \sigma_2 \neq 0,\ \sigma_3 \neq 0,\ \sigma_4 = 0$	$p^{4n}(p^n-1)^3$
$\sigma_1 \neq 0,\ \sigma_2 \neq 0,\ \sigma_3 = 0$	$p^{5n}(p^n-1)^2$
$\sigma_1 \neq 0,\ \sigma_2 = 0$	$p^{6n}(p^n-1)$
$\sigma_1 = 0$	p^{7n}

Subtracting the sum of the numbers in the last column from p^{8n}, the total number of sets $x_1, \ldots, x_8$, we obtain $p^n(p^n+1)(p^n-1)^6$ as the number of sets $x_1, \ldots, x_8$ for which (4) holds, and hence equals the order of the group.

$\xi_1, \ldots, \xi_8$ such that T replaces each η_i by a linear function of $\eta_1, \ldots, \eta_i$. The desired functions are

$$\eta_1=\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8, \quad \eta_2=\xi_1+\xi_2+\xi_3+\xi_4,$$
$$\eta_3=\xi_1+\xi_2+\xi_7+\xi_8, \quad \eta_4=\xi_1+\xi_2, \quad \eta_5=\xi_1+\xi_3+\xi_5+\xi_7,$$
$$\eta_6=\xi_1+\xi_3, \quad \eta_7=\xi_1+\xi_5, \quad \eta_8=\xi_1.$$

To prove them linearly independent modulo 2, we note that linear combinations of $\eta_1, \eta_2, \eta_3, \eta_4$ give $\xi_1+\xi_2, \xi_3+\xi_4, \xi_5+\xi_6, \xi_7+\xi_8$; while linear combinations of $\eta_5, \eta_6, \eta_7, \eta_8$ give $\xi_1, \xi_3, \xi_5, \xi_7$ We find that T gives rise to the following transformation:

$$\eta_1'=\rho_1\eta_1,$$
$$\eta_2'=\rho_2\eta_1+\rho_1\eta_2,$$
$$\eta_3'=\rho_3\eta_1 \quad +\rho_1\eta_3,$$
$$\eta_4'=\rho_4\eta_1+\rho_3\eta_2+\rho_2\eta_3 \quad +\rho_1\eta_4,$$
$$\eta_5'=\rho_5\eta_1+(\rho_2+\rho_3)\eta_2+\rho_3\eta_3 \quad +\rho_1\eta_5,$$
$$\eta_6'=\rho_6\eta_1+(\rho_3+\rho_4+\rho_5)\eta_2+\rho_4\eta_3 \quad +\rho_3\eta_4 \quad +\rho_1\eta_6,$$
$$\eta_7'=\rho_7\eta_1+(\rho_2+\rho_3+\rho_5)\eta_2+(\rho_3+\rho_4+\rho_5)\eta_3+\rho_2\eta_4 \quad +(\rho_2+\rho_3)\eta_5 \quad +\rho_1\eta_7,$$
$$\eta_8'=\rho_8\eta_1+(\rho_3+\rho_4+\rho_5+\rho_6+\rho_7)\eta_2+\rho_6\eta_3+(\rho_3+\rho_4+\rho_5)\eta_4+(\rho_2+\rho_4)\eta_5+(\rho_2+\rho_3)\eta_6+\rho_2\eta_7+\rho_1\eta_8,$$

the coefficients having the following values:

$$\rho_1=x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8, \quad \rho_2=x_5+x_6+x_7+x_8,$$
$$\rho_3=x_3+x_4+x_5+x_6, \quad \rho_4=x_5+x_6, \quad \rho_5=x_2+x_3+x_6+x_8,$$
$$\rho_6=x_6+x_8, \quad \rho_7=x_2+x_6, \quad \rho_8=x_6.$$

The determinant of the transformation is evidently ρ_1^8. The only condition on $\rho_1, \ldots, \rho_8$ is, therefore, that $\rho_1 \neq 0$. Hence the order of the group in the $GF[2^n]$ is $2^{7n}(2^n-1)$.

Since the expressions for $\eta_1', \eta_2', \eta_3', \eta_4'$ remain unaltered by the interchange of the variables η_i with the parameters ρ_i, we have a four-parameter *commutative subgroup* on the variables $\eta_1, \eta_2, \eta_3, \eta_4$.

A five-parameter commutative subgroup is obtained by setting $x_3=x_4$, $x_5=x_6$, $x_7=x_8$, so that $\rho_2=\rho_3=\rho_4\equiv 0 \pmod 2$. Also

$$\rho_1=x_1+x_2, \quad \rho_5=x_2+x_3+x_6+x_8, \quad \rho_6=x_6+x_8, \quad \rho_7=x_2+x_6, \quad \rho_8=x_6$$

are linearly independent functions of x_1, x_2, x_3, x_6, x_8 modulo 2. The above transformation now becomes

$$\eta_1' = \rho_1\eta_1, \quad \eta_5' = \rho_1\eta_5 + \rho_5\eta_1,$$
$$\eta_2' = \rho_1\eta_2, \quad \eta_6' = \rho_1\eta_6 + \rho_6\eta_1 \qquad + \rho_5\eta_2,$$
$$\eta_3' = \rho_1\eta_3, \quad \eta_7' = \rho_1\eta_7 + \rho_7\eta_1 \qquad + \rho_5\eta_3 + \rho_5\eta_3,$$
$$\eta_4' = \rho_1\eta_4, \quad \eta_8' = \rho_1\eta_8 + \rho_8\eta_1 + (\rho_5 + \rho_6 + \rho_7)\eta_2 + \rho_6\eta_3 + \rho_5\eta_4.$$

Denoting this transformation by T_ρ, we verify that $T_\rho\, T_{\rho'} = T_{\rho''}$, where

$$\rho_1'' = \rho_1\rho_1', \quad \rho_5'' = \rho_5\rho_1' + \rho_1\rho_5', \quad \rho_6'' = \rho_6\rho_1' + \rho_1\rho_6',$$
$$\rho_7'' = \rho_7\rho_1' + \rho_1\rho_7', \quad \rho_8'' = \rho_8\rho_1' + \rho_1\rho_8'.$$

Hence the transformations T_ρ form a group and the group is commutative. As this result is in accord with the general theory, we have a complete check upon the above work.

The University of Chicago, October 10, 1902.

THE LENGTH OF THE CIRCLE.

By DR. GEORGE BRUCE HALSTED.

The simple theorem of Euclid (Eu. I. 20) that any two sides of a triangle are together greater than the third, appears in popular geometries as a result, a deduction from the definition: the straight line is the shortest distance between two points. But such a definition is, as Hilbert says, *senseless* if one has not defined the concept length.

Without presupposing the idea of the length of the curve, it cannot use anything but the straight, and so is merely a *petitio principii*.

In fact, as Hilbert points out, the essential content of the statement, the straight line is the shortest distance between two points, reduces merely to this theorem of Euclid.

The same fallacy lurks in the theorem, "An arc of a circle is less than any line which envelops it and has the same extremities," recently 'borrowed' by Wentworth (1899, p. 219, Book V, Proposition VII, §451) from Chauvenet (1869, 1881, p. 155, Book V, Proposition XII, §32). He says "Of all the lines, etc., there must be at least one shortest line."

But what is the length of a curve? Again we have a begging of the question. Upon these two holes, empty places, or chasms, he then attempts to rest a demonstration that the length of a circle is the limit of the lengths of the perimeters of inscribed polygons.

The sooner this procedure is banished from elementary geometry the better for the world.

Fortunately there is here no need to touch that very difficult general question, what is the length of a curve in general.

We have only for consideration the simplest of curves, the circle.

For it then only need we show the simple theorem (Halsted's *Mensuration*, p. 16): When their sides tend indefinitely towards zero, the perimeter of the polygon inscribed increases, circumscribed decreases, toward the same limit.

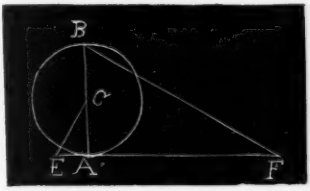
After that, we may deliberately and consciously declare by *definition* that this limit shall be what we will mean when we use the word *length* in connection with *circle*.

Here is no attempt to *prove* that the length of the circle is the limit of the lengths of perimeters of polygons. Any such attempt presupposes that we already know in some other way, or have in some other mathematical way defined what we are then already to mean by the length of the circle before we try to prove it equivalent to the limit for perimeters. What text book does this?

Can it be done? Let me try. (*Sect* is English for the German *strecke*).

Definition of length of an arc: We assume that with every arc is connected a sect such that if an arc be cut into two arcs, this sect is the sum of their sects; moreover this sect is not less than the chord of the arc, nor, if the arc be minor, is it greater than the sum of the sects on the tangents from the extremities of the arc to their intersection.

This sect we call *the length of the arc.*



Kochansky (1685) gave the following simple construction for the approximate length of the semicircle.

At the end-point A of the diameter BA, draw the tangent to the circle with center C. Take $ACE = \frac{1}{3}$ right angle. On the tangent, take $EF = 3AC$. Then BF is with great exactitude the length of the semicircle.

In fact, $BF = r[13\frac{1}{2} - 2(3)^{\frac{1}{2}}]^{\frac{1}{2}} = r3.1415$.

University of Texas, November, 1902.

AN ELEMENTARY ACCOUNT OF THE PICARD-VESSIOT THEORY.

By DR. SAUL EPSTEEN.

A theory of linear differential equations has been built up within the last twenty years which resembles the Galois theory of algebraic equations very closely.

In the Galois theory of equations we begin with an algebraic equation

$$(1) \qquad x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

and study the $n!$ permutations of the n roots $x_1, \ldots, x_n$.

In the corresponding theory of differential equations we begin with the linear homogeneous differential equation

$$(I) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + p_n y = 0$$

and study the transformations of the continuous group

$$y'_i = \sum_{k=1}^{n} a_{ik} y_k \quad (i = 1, \ldots, n),$$

where $y_1, \ldots, y_n$ form a fundamental system of integrals.

Many theorems analogous to well known theorems of algebra can be demonstrated in this new theory. For example, the theorem of Lagrange for indeterminate roots: "If a rational function $\phi(x_1, \ldots, x_n)$ remains unaltered by all the substitutions which leave another rational function $\psi(x_1, \ldots, x_n)$ unaltered, then ϕ can be expressed rationally in terms of ψ; $\phi = \text{Rat.}\ (\psi, a_1, \ldots, a_n)$ has the following analog: If a rational function $\phi(y_1, \ldots, y_n)$ of the integrals remains invariant under all the transformations which constitute the group of another rational function $\psi(y_1, \ldots, y_n)$ then ϕ is rationally expressible in terms of ψ; $\phi = \text{Rat.}\ (\psi, p_1, \ldots, p'_1, \ldots, x)$."

As is well known, the group of an algebraic equation is defined by the two fundamental properties:

(a) Every rational function of the roots, which remains unaltered by all the substitutions of the group, is rationally known.

(b) Every rational function of the roots, which is rationally known, remains unaltered by all the substitutions of the group.

The parallel defining properties of the group of a differential equation are:

(A) Every rational function of the integrals (and their derivatives), which remains unaltered by all the transformations of the group, is rationally known.

(B) Every rational function of the integrals (and their derivatives), which is rationally known, remains unaltered by all the transformations of the group.

This brings us now to the question of formal and numerical invariance which is very much alike in both theories. The question will be illustrated by examples.

Let the algebraic equation be $x^4 + 1 = 0$ with the roots

$$x_1 = \varepsilon, \ x_2 = i\varepsilon, \ x_3 = -\varepsilon, \ x_4 = -i\varepsilon \quad \left(\varepsilon = \frac{1+i}{\sqrt{2}}\right).$$

The group of this equation is*

$$G = [\mathrm{I}; \ (x_1 x_2)(x_3 x_4); \ (x_1 x_3)(x_2 x_4); \ (x_1 x_4)(x_2 x_3)].$$

*Bolza, *Bulletin American Mathematical Society*, First Series, Vol. 2, p. 99.

For the domain of rational numbers, the function x_1x_2 has the rational value -1. The substitutions of G which leave x_1x_2 formally invariant are $[I; (x_1x_2)(x_3x_4)]$, which form only a subgroup of the group of the equation. But all the substitutions of G leave x_1x_2 numerically invariant, since $x_1x_2=x_3x_4=-1$.*

Likewise in differential equations. The integral being regarded as functions of x, namely, $y_i=y_i(x)$ $(i=1, 2, \ldots, n)$, we consider, not the continuous group which leaves some function such as $\varphi(y_1, \ldots, y_n)=r(x)$ invariant as a function of the y's, but we consider a group which may change $\varphi(y_1, \ldots, y_n)$ to another function $\psi(y_1, \ldots, y_n)$ provided that *the numerical value is unaltered:* $\varphi=\psi=r(x)$. The value of x is not specified.

As an illustration, consider an irreducible linear homogeneous differential equation of the third order,

$$\frac{d^3y}{dx^3}+p_1\frac{d^2y}{dx^2}+p_2\frac{dy}{dx}+p_3y=0,$$

between a fundamental set of whose integrals there exists the *simple* rational relation

$$\varphi(y_1, y_2, y_3)\equiv y_2^2-y_1y_3=0.$$

If there were several such relations (as may well happen for equations of higher order) we must consider the group which leaves them all simultaneously invariant numerically.

The 3-parameter group

$$(2)\qquad \begin{aligned} y_1'&=a^2y_1+2aby_2+b^2y_3\\ y_2'&=acy_1+(ad+bc)y_2+bdy_3\\ y_3'&=c^2y_1+2cdy_2+d^3y_3 \end{aligned}$$

with $ad-bc=1$, leaves φ formally invariant since

$$y_2'^2-y_1'y_3'\equiv y_2^2-y_1y_3.$$

But it is not the group of the equation. The group of the equation (in the domain of the coefficients) is the 4-parameter group (2) where $ad-bc=$any constant different from zero. This latter group does not leave φ formally invariant; for now

$$y_2'^2-y_1'y_3'=(ad-bc)^3(y_2^2-y_1y_3).$$

*From this it must not be concluded that all the substitutions which leave x_1x_2 numerically unchanged: $G_8=[I; (x_1x_2); (x_3x_4); (x_1x_2)(x_3x_4); (x_1x_3)(x_2x_4); (x_1x_4)(x_2x_3); (x_1x_3x_2x_4); (x_1x_4x_2x_3)]$ form the group of the equation. The reason for this lies in (*b*); for we must consider not merely G_8 which leaves the *single* function $x_1x_4=-1$ numerically invariant, but we must consider a group (which turns out to be G) which leaves *every* function which has a rational value, unaltered. Thus $x_1x_2^2+x_3x_4^2=0$ has a rational value. Now the transposition (x_1x_3) of G_8 changes this to $x_3x_1^2+x_3x_4^2=2(i-1)$, therefore G_8 is not the group of the equation. Notice that the group of the equation, G, will actually leave $x_1x_2^2+x_3x_4^2=0$ numerically unaltered.

The numerical value zero (for $\varphi=0$) is however unaltered by the group of the equation. (If φ were equal to $r(x)$ instead of to zero, the group of the equation would be (2) with $ad-bc=1$; for $ad-bc\neq 1$, the numerical value would change from $r(x)$ to $(ad-bc)^3 r(x)$.)

In the above example we assumed that the function $\varphi=y_2^2-y_1y_3$ was the invariant of the group. If the differential equation was of a higher order than the third the relation would be more complicated. Indeed, we might have a number of such relations involving not only the integrals but also their derivatives, say $\varphi_i(y_1....y_n, y_1'....y_n, y_1''....)$, $(i=1....p)$. The generalization can be carried still further. Notice that we assumed above that there exists algebraic relations between the integrals and their derivatives. Now this assumption is unnecessary, it may well happen that there are no such relations whatever, yet we continue still to speak of the group of the equation. An exposition of this question would be beyond the range set for this paper.

The group in question is technically called the *group of rationality* of the equation.

The literature of this subject is now quite extensive, but anyone familiar with the elements of the Galois theory will find it very easy to read the brief summary in Picard's *Traité de Analyse* III, last chapter, and to supplement it with Klein's *Höhere Geometrie* II, pp. 298-9.

The University of Chicago, October, 1902.

TWO SIMPLE CONSTRUCTIONS FOR FINDING THE FOCI OF AN HYPERBOLA, GIVEN THE ASYMPTOTES AND A POINT ON, OR A TANGENT TO, THE CURVE.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina.

The construction for a special position of the point is given first, as it is a linear construction.

Given the asymptotes of a hyperbola and the vertex A of the curve, to construct the foci.

The major axis is fixed, bisecting the angle between the asymptotes. Lay off, along an asymptote, from the center C of the hyperbola a distance $CD=CA=a$; draw a perpendicular to the asymptote at D. This meets the major axis (produced) in a focus $F_1(F_2)$.

For, the perpendicular F_1D from the focus $(ae, 0)$ upon the asymptote, whose equation is $y=\frac{b}{a}x$, is equal to b (by elementary principles). But $CF_1=\sqrt{(a^2+b^2)}$ and therefore $CD=a$.

The following style of argument (communicated to me in a letter a few

days ago), while ordinarily to be avoided, is rather interesting in this connection. Granting a previous knowledge of the property in question (such as given by the above proof), this second proof furnishes an example of the use of the indeterminate form $\infty - \infty = k$ (k=constant), in a geometrical problem.

Let P, a point on the curve, move out to infinity on the upper half of the right-hand branch of the hyperbola. From F_1, the right-hand focus, draw a perpendicular to CP_∞, meeting it at D and meeting F_2P_∞ at E.

Now, by the defining property of the hyperbola,

$$F_2P_\infty - F_1P_\infty = 2a.$$

Therefore $F_2E = 2a$, and consequently $CD = a$.

The general problem is the following: *Given the asymptotes of an hyperbola and a point P on, or a tangent DE to, the curve; to construct the foci.*

If we are given P, DE is determined; for, PG, drawn parallel to one (either) asymptote, meets the other at G, say. GE laid off equal to OG fixes the tangent DE in position, since the segment of any tangent to an hyperbola intercepted between the asymptotes is bisected at the point of contact.

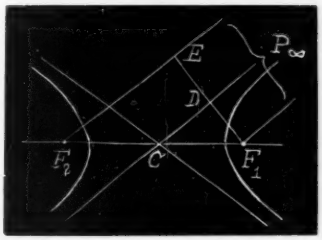


Fig. 1.

Hence only the case in which the tangent is given will be considered, since the other may be immediately reduced to it.

Construction. Draw PC perpendicular to the tangent DE at its middle point P, and let it intersect the minor axis at C. Then the circle with center C and radius $CD(=CE)$ cuts the major axis in the two foci F_1 and F_2.

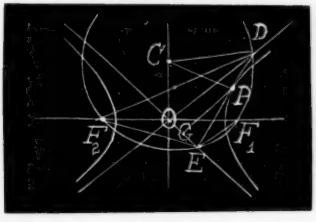


Fig. 2.

For, if P, D, C are designated respectively by (x_1, y_1), (x_2, y_2), $(0, y_3)$, we find by obvious methods that

$$y_3 = \frac{a^2e^2y_1}{b^2}, \quad x_2 = \frac{bx_1 + ay_1}{b}, \quad y_2 = \frac{bx_1 + ay_1}{a},$$

and therefore, by the formula for the distance between two points, we obtain $CD = CF_1$.

This may be very easily proved by projective geometry in the following manner. It is a well known property that the angle $EF_1D(EF_2D)$ is invariant for all positions of the variable tangent DE. Hence, when ED coincides with OD, $\angle EF_1D = \angle DOF_2$, $\angle EF_2D = \angle DOF_1$ (since when D moves out to infinity, F_1D_∞ and F_2D_∞ are parallel to OD. Accordingly,

$$\angle EF_1D + \angle EF_2D = 180^\circ$$

and the points F_1, E, F_2, D are concyclic. The perpendicular bisectors of F_1F_2 and DE therefore intersect at C, the required center.

It may be observed that the second construction is a direct generalization of the one (when the asymptotes and vertex are given) found in all elementary text-books on Analytic Geometry. Would it not be well for the latter to contain the above simple constructions?

The University of Chicago, October, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

161. Proposed by F. M. SHIELDS, Coopwood, Miss.

If 1 man, 1 boy, and 1 girl catch 1 trout, 1 perch, and one minnow in 5 minutes, and 1 man, 2 boys, and 3 girls catch 1 trout, 2 perch, and 3 minnows in 6 minutes, how many minutes will be required for 2 men, 3 boys, and 4 girls to catch 5 trout, 11 perch, and 17 minnows?

Remarks by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

This problem is indeterminate. Let A_1, A_2, and A_3 be the activities of a man, boy, and girl, respectively; and E_1, E_2, and E_3 the work required to catch a trout, a perch, and a minnow, respectively. Then

$$5(A_1+A_2+A_3)=E_1+E_2+E_3....(1), \text{ and}$$
$$6(A_1+2A_2+3A_3)=E_1+2E_2+3E_3....(2).$$

From these two conditions it is required to find t so that the following condition is satisfied, viz:

$$t(2A_1+3A_2+4A_3)=5E_1+11E_2+17E_3.$$

This is clearly impossible in general. By assuming certain other conditions as, for example, the ratios of the activities, and the ratios of the work, a solution may be effected.

162. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A trolly road is built between two towns, and it is found that the gross annual receipts amount to 20% of the original cost; the annual cost of repairs 2% of the original cost; and the working expense is \$3000 in addition to 20% of the net receipts. After a year a second road is built at the same cost as the first and it is found that the gross re-

ceipts and working expenses per year are doubled, while the cost of repairs for the new road is 1% of cost. If the receipts for both roads is \$72,500, find the cost of each road, and the net receipts the first year.

Solution by the PROPOSER.

Let 100% = cost of each road, 20% = gross receipts first year, 2% = repairs first year.

Working expenses = \$3000 + 20% of (20% − 2% − working expenses).

$\therefore$ $\frac{6}{5}$ working expenses = \$3000 + 3.6%.

$\therefore$ working expenses = \$2500 + 3%.

The second year, gross receipts = 40%, repairs = 3%, working expenses = \$5000 + 6%.

$\therefore$ \$72,500 = 40% − 3% − 6% − \$5000.

$\therefore$ 31% = \$77,500, 100% = \$250,000, cost of each road.

Net receipts first year = 20% − 2% − \$2500 − 3% = 15% − \$2500 = \$35,000.

Also solved by *W. A. CLEMENSER*, Heidelberg University, Tiffin, O.; and *C. A. SHORT*, Assistant Professor of Mathematics, Delaware College, Delaware.

ALGEBRA.

147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove that $x = a^x$ has never more than two real roots, and find the condition for no real roots.

Solution by the PROPOSER.

$x = a^x$. $\therefore$ $a = x^{1/x}$. $a_1 = \frac{x^{1/x}}{x^2}(1 - \log x)$ where $a_1 = \frac{da}{dx}$ and $a_1 = 0$ for $x = 0, e, \infty$. For $x > 1$, a is > 1; for $0 < x < 1$, $0 < a < 1$, for $x = 1$, $a_1 = 1$.

$\therefore$ The curve represented by the equation touches axis of x at $x = 0$, and its ordinates increase from 0 to 1 with its abscissæ; at $x = 1$ it is touching the bisector of the axes. From 1 to e for x, ordinates increase from 1 to $e^{1/e}$, and as x increases from e to ∞, the ordinates decrease from $e^{1/e}$ to 1.

$\therefore$ For $0 < a < 1$, a real root between 0 and 1,

$1 < a < e^{1/e}$, a real root between 1 and $e^{1/e}$, or another between e and ∞.

$a = e^{1/e}$, the root is e.

If $a > 1$ and $re^{\theta i}$ an imaginary root, then if $r\cos\theta = u$, $r\sin\theta = v$ we have $re^{\theta i} = e^{\log a re^{\theta i}} = e^{\log(a + \theta i)}$ and the equation becomes, on equating real and imaginary parts, $\log r = u\log a$; $\theta = v\log a$. The intersection of these curves gives the imaginary roots required, but we must reject roots between the lines $v\log a = (2k+1)\pi$ and $v\log a = (2k+2)\pi$. If $a < 1 = 1/b$ the curves are $r = b^{-u}$, $\theta = -v\log b$, and these curves are the images of the two previous, and the branches giving the roots of the equation proposed are the images of the branches which we previously rejected, and vice versa.

For a full discussion of this equation see Nouville's *Annales*, 1896, p. 548; 1897, p. 54.

158. Proposed by R. D. BOHANNON, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show,

$$\frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$$

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Consider the equation,

$$\frac{x}{a+\varphi}+\frac{y}{b+\varphi}+\frac{z}{c+\varphi}=1-\frac{(\varphi-\alpha)(\varphi-\beta)(\varphi-\gamma)}{(a+\varphi)(b+\varphi)(c+\varphi)}.$$

Multiply through by $a+\varphi$ and then put $a+\varphi=0$.

$$\therefore x=\frac{(a+\alpha)(a+\beta)(a+\gamma)}{(a-b)(a-c)},\quad y=\frac{(b+\alpha)(b+\beta)(b+\gamma)}{(b-c)(b-a)},\quad z=\frac{(c+\alpha)(c+\beta)(c+\gamma)}{(c-a)(c-b)}$$

$$\therefore \frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(a+\alpha)(a+\gamma)}{(a+\beta)(a-b)(a-c)}$$

$$+\frac{(b+\alpha)(b+\gamma)}{(b+\beta)(b-c)(b-a)}+\frac{(c+\alpha)(c+\gamma)}{(c+\beta)(c-a)(c-b)}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$$

Also $$\frac{x}{(a+\alpha)^2}+\frac{y}{(b+\alpha)^2}+\frac{z}{(c+\alpha)^2}=\frac{(\gamma-\alpha)(\beta-\alpha)}{(a+\alpha)(b+\alpha)(c+\alpha)}.$$

$$\frac{x}{(a+\gamma)^2}+\frac{y}{(b+\gamma)^2}+\frac{z}{(c+\gamma)^2}=\frac{(\alpha-\gamma)(\beta-\gamma)}{(a+\gamma)(b+\gamma)(c+\gamma)}.$$

Also solved by *LON C. WALKER*.

159. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x-1=3m$, $x^2-1=4n$, $x^3-1=5p$, where m, n, p are integers, find a general expression for x.

I. Solution by L. E. DICKSON, Ph. D., The University of Chicago, Chicago, Ill.

The required expression for x is of the form $3m+1$, where m is subject to the conditions that $(3m+1)^2-1$ shall be divisible by 4, and that $(3m+1)^3-1$ shall be divisible by 5. The first condition is satisfied if, and only if, m is even. The second condition requires that either m or else $3m^2+3m+1$ shall be divisible by 5. To prove that the last alternative is impossible, we note that

$$2(3m^2+3m+1)\equiv(m+3)^2-2 \pmod 5,$$

and hence is not$\equiv 0 \pmod 5$, 2 being a non-quadratic residue of 5. The necessary and sufficient conditions are, therefore, that m shall be divisible by 10, so that x shall have the form $x=30t+1$.

II. Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

From the three equations we readily obtain

$$\begin{aligned} x-1&=3m &&=3m,\\ x^2-1&=3m(2+3m) &&=4n,\\ x^3-1&=9m[1+3m(1+m)]&&=5p; \end{aligned}$$

whence $n=\frac{3}{4}m(2+3m)$ and $p=\frac{9}{5}m[1+3m(1+m)]$,

where m must be so taken as to make n and p integers. To make n integral m must be an *even* number; to make p integral m must be a multiple of 5; hence $m=10k$ where k may be any positive integer. Hence

$$\begin{aligned} m&=10k,\\ n&=15k(1+15k),\\ p&=18k[1+30k(1+10k)]. \end{aligned}$$

Also
$$\begin{aligned} x-1&=30k,\\ x^2-1&=60k(1+15k),\\ x^3-1&=90k[1+30k(1+10k)]; \end{aligned}$$

and $x=1, 31, 61, 91$, etc.

Solved similarly by *G. B. M. ZERR*, *LON C. WALKER*, and *HON. J. H. DRUMMOND*.

160. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Represent the square root of $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$ as the sum of three square roots.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University; and the late HON. JOSIAH H. DRUMMOND.

Let $\sqrt{[10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}]}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

$\therefore\ 10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}$.

$\therefore\ 2\sqrt{xy}=2\sqrt{6}$, $\therefore\ \sqrt{xy}=\sqrt{6}$. Similarly, $\sqrt{xz}=\sqrt{10}$, $\sqrt{yz}=\sqrt{15}$.

$\therefore\ \sqrt{(xyz)}=\sqrt{30}$. $\therefore\ \sqrt{x}=\sqrt{2}$, $\sqrt{y}=\sqrt{3}$, $\sqrt{z}=\sqrt{5}$.

$\therefore\ \sqrt{[10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}]}=\sqrt{2}+\sqrt{3}+\sqrt{5}$=the sum of the three square roots.

GEOMETRY.

187. Proposed by G. TUCKER, M. A.

AD, BE, CF are the altitudes of the triangle ABC; k_1, k_1'; k_2, k_2'; k_3, k_3' are the S points of the triangles EAB, FCA; FBC, DCB; DCA, EBC, respectively; prove that $k_3'k_1=k_1'k_2=k_2'k_3=R\sin A\sin B\sin C$. ρ_1, ρ_1'; ρ_2, ρ_2'; $\rho_3\rho_3'$ are the Brocard radii of the above triangles, prove that (1) $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$; (2) $(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\frac{3}{64}$; (3) the sets of 4 Brocard-points for the above pairs of triangles are concyclic (on three circles); (4) the tangents from any one of the right angles of the above triangles to the Brocard circle of the tangent is a mean proportional between the tangents to the same circle from the remaining (two) angles.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The S point of each triangle is found by drawing from its right angle a line perpendicular to its opposite side, and taking its mid-point. Let k_1, k_2, and k_3 be the mid-points of HF, FK, and DI, respectively, and k_1', k_2', k_3', the mid-points of GE, LD, and JE, respectively. From the inscribed quadrilateral $BGEJ$ we get $GB.JE+BJ.GE=GJ.BE$, $JE\sin A+GE\sin C=GJ$.

$\therefore$ $\sin A\sin C(a\cos C+c\cos A)=GJ=b\sin A\sin C$.

Similarly, $KH=c\sin A\sin B$, $LI=a\sin B\sin C$.

$\therefore$ $GJ=KH=LI=2R\sin A\sin B\sin C$.

$\therefore$ $k_3'k_1=\frac{1}{2}GJ=k_1'k_2=\frac{1}{2}KH=k_2'k_3=\frac{1}{2}LI=R\sin A\sin B\sin C$.

(1) Let $\omega_1, \omega_1', \omega_2, \omega_2', \omega_3, \omega_3'$ be the Brocard angles; then $\omega_1=\omega_1'$, $\omega_2=\omega_2'$, $\omega_3=\omega_3'$, for $\cot\omega_1=\cot\omega_1'=\cot A+\tan A$.

$\therefore$ $b\rho_1=c\rho_1'=\frac{1}{4}bc(1-3\cos^2 A\sin^2 A)^{\frac{1}{2}}$, $c\rho_2=a\rho_2'=\frac{1}{4}ac(1-3\cos^2 B\sin^2 B)^{\frac{1}{2}}$, $a\rho_3=b\rho_3'=\frac{1}{4}ab(1-3\cos^2 C\sin^2 C)^{\frac{1}{2}}$.

$\therefore$ $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$, $(\rho_2'^2-\rho_3^2)a^2+(\rho_3'^2-\rho_1^2)b^2+(\rho_1'^2-\rho_2^2)c^2=0$.

(2) $(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\dfrac{c^2-b^2}{16a^2}+\dfrac{a^2-c^2}{16b^2}$

$+\dfrac{b^2-a^2}{16c^2}-\dfrac{3\triangle^4}{a^2b^2b^2}\left[\dfrac{c^2-b^2}{a^4}+\dfrac{a^2-c^2}{b^4}+\dfrac{b^2-a^2}{c^4}\right]$.

(3) Let M, M_1 be the Brocard-points of EAB; M_2, M_3 those of FCA. Since $\omega_1=\omega_1'$, AM_3M and AM_1M_2 are three each on a straight line.

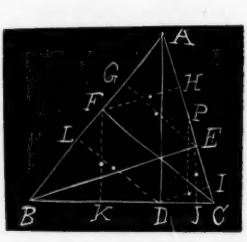
But $AM=c\sin\omega_1/\sin A$, $AM_3=b\sin\omega_1\cot A$,

$AM_2=b\sin\omega_1/\sin A$, $AM_1=c\sin\omega_1\cot A$.

$\therefore$ $AM.AM_3=AM_1.AM_2$ or $AM:AM_2=AM_1:AM_3$.

$\therefore$ M, M_1, M_2, M_3 are concyclic.

Similarly for the remaining pairs of triangles.



(4) Let t, t_1, t_2 be the tangents from D, C, A to the Brocard circle.

Then $t=(DI.Dk_3)^{\frac{1}{2}}=\dfrac{1}{\sqrt{2}}b\sin\frac{1}{2}C\cos\frac{1}{2}C$,

$$t_1=(CP.CI)^{\frac{1}{2}}=\frac{1}{\sqrt{2}}b\cos C,\quad t_2=(AP.AI)^{\frac{1}{2}}=\frac{1}{\sqrt{2}}b\sin C.$$

$\therefore t^2=t_1t_2.\quad \therefore t_1:t=t:t_2.$

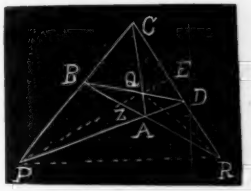
Similarly for the other triangles.

Also proved by *J. R. HITT.*

188. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

ABCD is a quadrilateral whose diagonal triangle is *PQR*, *P* on *AD* and *R* on *AB*. *PQ* meets *AB* in *Z*. If *C* moves along *PB* what will happen to *Z*?

Solution by J. R. HITT, A. M., Coronal Institute, San Marcos, Tex., and G. W. GREENWOOD, A. M., McKendree College, Lebanon, Ill.



Let the figure be drawn as indicated in the proposition. Now, the sides *PQ*, *PR*, of the diagonal triangle, and the opposite sides *AD*, *BC*, of the quadrilateral, form a harmonic pencil. The three rays, *PR*, *PD*, *PC*, remain fixed as *C* moves along *BC*; hence, ray *PE* remains fixed. Therefore, *Z* is fixed, since *RB* is fixed.

Also demonstrated by *G. B. M. ZERR.*

189. Proposed by J. C. CORBIN, Pine Bluff, Ark.

The perpendicular from the right angle on the hypotenuse of a right-angled-triangle is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle. [From Casey's *Sequel to Euclid.*]

Solution by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, and J. E. SANDERS, Hackney, Ohio.

Denote by *ABC* the right-angled-triangle, whose sides opposite the angles *A*, *B*, *C* are *a*, *b*, *c*, respectively, where *C* is the right angle.

Let *r* be the radius of the incircle with center *O*, which is tangent to *AB* at the point *G*. Draw *CD* perpendicular to *AB*, *OE* to *BC*, and *OF* to *AC*. Then by the condition of the problem, we have

$$CD=\frac{2BG.GA}{BG+GA}\ldots.(1).$$

Since $AG=AF=b-r=\frac{1}{2}[c-(a-b)]$, and $BG=BE=a-r=\frac{1}{2}[c+(a-b)]$, the right member of (1) reduces to

$$\frac{c^2-(a-b)^2}{2c}=\frac{ab}{c}\ldots.(2).$$

Now from the similar right triangles *BDC* and *BCA*, we have

$$CD:AC=BC:AB, \text{ or } CD:a=b:c,$$

from which $CD=ab/c$, which agrees with result (2).

Also solved by *J. R. HITT, J. SCHEFFER*, and *G. B. M. ZERR*.

CALCULUS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find volume common to the two solids

$$[x/a]^{\frac{1}{2}} + [y/b]^{\frac{1}{2}} = [z/c]^{\frac{1}{2}}, \quad [y/b]^{\frac{1}{2}} + [z/c]^{\frac{1}{2}} = [x/a]^{\frac{1}{2}}.$$

Solution by the PROPOSER.

From the equations $[x/a]^{\frac{1}{2}} + [y/b]^{\frac{1}{2}} = [z/c]^{\frac{1}{2}}$ and $[y/b]^{\frac{1}{2}} + [z/c]^{\frac{1}{2}} = [x/a]^{\frac{1}{2}}$ we find the limits of z to be $z=c\{[x/a]^{\frac{1}{2}} + [y/b]^{\frac{1}{2}}\}^2$ to $z=c\{[x/a]^{\frac{1}{2}} - [y/b]^{\frac{1}{2}}\}^2$. Eliminating z we get

$$y=\pm\frac{b}{\sqrt{8}}\left[\frac{x}{a}\right]^{\frac{1}{2}}\{1-[x/a]^{\frac{1}{2}}\}^2 = y'.$$

The limits of x are $x=0$ to $x=a$.

$$\therefore V=2c\int_0^a \int_0^{y'}[\{(x/a)^{\frac{1}{2}} - (y/b)^{\frac{1}{2}}\}^2 - \{(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}}\}^2]dxdy.$$

Let $x/a=u^3$, $y/b=v^3$.

$$\therefore V=18abc\int_0^1 \int_0^{\sqrt{[\frac{1}{4}u(1-u)]}} u^2v^2\{(u-v^2)^{\frac{1}{2}} - (u^2+v^2)^{\frac{1}{2}}\}dudv$$

$$=\tfrac{3}{8}abc\int_0^1 \Big(u^2(2+u+2u^2)\sqrt{(1-u^2)}-8(1-u^2)^{\frac{3}{2}}+3u^2\sin^{-1}\left[\frac{1-u}{2}\right]$$

$$+6u^5\log\left[\frac{1+\sqrt{(1-u^2)}}{u}\right]\Big)u^3du$$

Let $u=\cos\theta$.

$$\therefore V=\tfrac{3}{16}abc\int_0^{\frac{1}{2}\pi} \Big(4\cos^2\theta\sin\theta+2\cos^3\theta\sin\theta+4\cos^4\theta\sin\theta-16\sin^5\theta+3\theta\cos^2\theta$$

$$+12\cos^5\theta\log\left[\frac{1+\sin\theta}{\cos\theta}\right]\Big)\cos^3\theta\sin\theta d\theta=\frac{115\pi abc}{2048}.$$

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Helmholtz's differential equation for the strength of an electric current C at any time t, is $C=E/R-L/R\times dC/dt$. Solve this equation, supposing $C=0$ when $t=0$; and E, R, L are to be regarded as constants.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

The equation can be written $LdC=(E-CR)dt$.

$$\therefore \frac{LdC}{E-CR}=dt, \text{ or } \frac{L}{R}\log(E-CR)+t+B=0.$$

When $C=0$, $t=0$, and $B=-\frac{L}{R}\log E$.

$$\therefore t+\frac{L}{R}\log(E-CR)=\frac{L}{R}\log E, \text{ or } t+\frac{L}{R}\log\left(\frac{E-CR}{E}\right)=0.$$

149. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume contained between the plane $z=(a-x)\cot\beta$ and the surface $xz^2=(a-x)(x^2-y^2)$.

Solution by the PROPOSER.

The z limits are $z=\sqrt{\frac{a-x}{x}\ (x^2+y^2)}$ to $z=(a-x)\cot\beta$.

Eliminating z, $y=\pm\sqrt{[x(a\cot^2\beta-x\operatorname{cosec}^2\beta)]}=y'$.
The x limits are 0 and $a\cos^2\beta=x'$.

$$V=2\int_0^{x'}\int_0^{y'}\left[(a-x)\cot\beta-\sqrt{\frac{a-x}{x}\ (x^2+y^2)}\ \right]dxdy$$

$$=\int_0^{x'}\left[\sqrt{x}\ (c-x)\cot\beta\sqrt{[a\cot^2\beta-x\operatorname{cosec}^2\beta]}\right.$$

$$\left.-x\sqrt{(ax-x^2)}\log\left(\frac{\sqrt{(a-v)}\cot\beta+\sqrt{(a\cot^2\beta-x\operatorname{cosec}^2\beta)}}{\sqrt{x}}\right)\right]dx.$$

Let $x=a\cos^2\beta\sin^2\theta$.

$$\therefore V=2a^3\cos^3\beta\cot^2\beta\int_0^{\frac{1}{2}\pi}(1-\cos^2\beta\sin^2\theta)\sin^2\theta\cos^2\theta d\theta$$

$$-2a^3\cos^5\beta\int_0^{\frac{1}{2}\pi}\sqrt{(1-\cos^2\beta\sin^2\theta)}\sin^4\theta\cos\theta\log\left(\frac{\sqrt{(1-\cos^2\beta\sin^2\theta)}+\cos\theta}{\sin\beta\sin\theta}\right)d\theta.$$

Let $\cos\beta\sin\theta=\sin\varphi$; then the second term becomes

$$2a^3\int_0^{\frac{1}{2}\pi-\beta}\sin^4\varphi\cos^2\varphi\log\left[\frac{\cos\varphi\cos\beta+\sqrt{(\cos^2\beta-\sin^2\varphi)}}{\sin\beta\sin\varphi}\right]d\varphi=2a^3\int_0^{\frac{1}{2}\pi-\beta}(\tfrac{1}{16}\varphi-$$

$$\tfrac{1}{16}\sin\varphi\cos\varphi-\tfrac{1}{24}\sin^3\varphi\cos\varphi+\tfrac{1}{6}\sin^5\varphi\cos\varphi)\frac{\cos\beta d\varphi}{\sin\varphi\sqrt{(\cos^2\beta-\sin^2\varphi)}}.$$

$$\therefore\ V=\tfrac{1}{16}\pi a^3\cos^3\beta\cot^2\beta(1+\sin^2\beta)+\tfrac{1}{8}a^3\cos\beta\int_0^{\frac{1}{2}\pi}d\theta+\tfrac{1}{12}\cos^3\beta\int_0^{\frac{1}{2}\pi}\sin^2\theta d\theta$$

$$-\tfrac{1}{8}a^3\cos^5\beta\int_0^{\frac{1}{2}\pi}\sin^4\theta d\theta-\tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}.$$

$$\therefore\ V=\tfrac{1}{16}\pi a^3\cot\beta\operatorname{cosec}\beta-\tfrac{1}{24}\pi a^3\cos^3\beta$$

$$-\tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}.\quad\tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}$$

$$=\tfrac{1}{8}a^3\cos\beta\int_0^{\frac{1}{2}\pi}(1+\tfrac{2}{3}\cos^2\beta\sin^2\theta+\tfrac{8}{15}\cos^4\beta\sin^4\theta+\tfrac{16}{35}\cos^6\beta\sin^6\theta+\ldots.)d\theta$$

$$=\tfrac{1}{16}\pi a^3(\cos\beta+\tfrac{1}{3}\cos^3\beta+\tfrac{1}{5}\cos^5\beta+\tfrac{1}{7}\cos^7\beta+\ldots.)=\tfrac{1}{16}\pi a^3\log\cot\tfrac{1}{2}\beta.$$

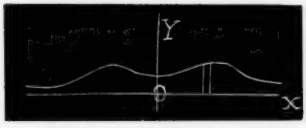
$$\therefore\ V=\tfrac{1}{48}\pi a^3(3\cot\beta\operatorname{cosec}\beta-2\cos^3\beta-3\log\cot\tfrac{1}{2}\beta).$$

150. Proposed by E. B. ESCOTT, Instructor in Mathematics, University of Michigan, Ann Arbor, Mich.

Find total area between the curve $x^4y-x^2+4y-1=0$ and the x-axis.

Solution by J. E. SANDERS, Hackney, Ohio, and the PROPOSER.

The equation may be written $y=\dfrac{x^2+1}{x^4+4}$.



$$\text{Area}=2\int_0^\infty y\,dx=2\int_0^\infty\frac{x^2+1}{x^4+4}dx$$

$$=\tfrac{1}{4}\left[\int_0^\infty\frac{x+2}{x^2-2x+2}dx-\int_0^\infty\frac{x-2}{x^2+2x+2}dx\right]=\tfrac{1}{8}\Big[\log(x^2-2x+2)$$

$$-\log(x^2+2x+2)+6[\tan^{-1}(x+1)+\tan^{-1}(x-1)]\Big]_0^\infty$$

$$=\tfrac{1}{8}\log\frac{x^2-2x+2}{x^2+2x+2}+6\tan^{-1}\frac{2x}{2-x^2}\Big]_0^\infty=\tfrac{1}{8}(6\pi)=\tfrac{3}{4}\pi.\quad\text{Answer.}$$

$\left[\tan^{-1}\frac{2x}{2-x^2}\right]_{x=\infty}=\tan^{-1}0=\pi$, since the integrand has increased and passed through ∞ for $x=\sqrt{2}$.

Also solved by *G. B. M. ZERR.*

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\frac{2\pi y}{a}\right)+\sin h\left(\frac{2\pi y}{a}\right)$.

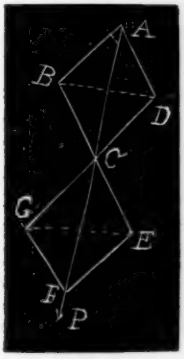
No solution of this problem has been received.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

$ABCD$, $GCEF$ are equal parallelograms, DCG and BCE being straight lines. If the figure be considered as formed of smooth light jointed bars and if BD be a light rod, and the whole be suspended from A, find the stress in BD if a weight be hung from F. Also find the stress if a light rod GE replace BD.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Since the bars are light we can disregard their weight. Let P be the weight. Then by virtual work



$$Pd(AC)+Sd(BD)=0....(1).$$

But $AC^2+BD^2=2AD^2+2DC^2$.

$\therefore ACd(AC)+BDd(BD)=0....(2)$.

From (1) and (2),

$$\frac{d(AC)}{d(BD)}=-\frac{S}{P}=-\frac{BD}{AC}.$$

$\therefore S=\frac{P.BD}{AC}$. Similarly for GE.

$S_1=\frac{P.GE}{CF}$. The stress is the same for both.

146. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A diffraction grating, with lines .05 mm. apart is held in front of a Bunsen's burner in which common salt is volatilized, and, when viewed through a telescope it is found that the angular distances of the first, second, third, fourth, fifth, and sixth bright bands from the central one are respectively 41′, 1°21′, 2°2′, 2°42′, 3°23′ and 4°3′. Required the wave length of sodium light.

Solution by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio, and the PROPOSER.

Let λ=wave length, d=distance apart of grating, θ=angular distance, n=order from central one. Then $\lambda=d\sin\theta/n$.

$$\frac{.05\sin 41'}{1}=.000596305\text{mm.},\quad \frac{.05\sin 1°21'}{2}=.000588995\text{mm.},$$

$$\frac{.05\sin 2°2'}{3}=.000591348\text{mm.},\quad \frac{.05\sin 2°42'}{4}=.000588831\text{mm.},$$

$$\frac{.05\sin 3°21'}{5}=.000590160\text{mm.},\quad \frac{.05\sin 4°3'}{6}=.000588558\text{mm.}$$

λ=the mean of these six. $\therefore\ \lambda=.0005906995$mm.

DIOPHANTINE ANALYSIS.

99. Proposed by the late HON. JOSIAH H. DRUMMOND, LL. D.

If p and q are such values of x and y as fulfill the conditions $x^2\pm y^2-1$=a square, find, in terms of p and q, the expression for an indefinite number of other values.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$x^2\pm y^2=a^2+1=b,\text{ or }\frac{x^2}{b}\pm\frac{y^2}{b}=1.$$

Let $x=p$, and $y=q$ be a solution. Then $(p^2/b\pm q^2/b)^n=1$.

$\therefore\ [x/\sqrt{b}+y_1\sqrt{(\pm 1)/b}][x/\sqrt{b}-y_1\sqrt{(\pm 1)/\sqrt{b}}]=[p/\sqrt{b}+q_1\sqrt{(\pm 1)/\sqrt{b}}]^n \times[p/\sqrt{b}-q_1\sqrt{(\pm 1)/\sqrt{b}}]^n$.

Put $x/\sqrt{b}+y_1\sqrt{(\pm 1)/\sqrt{b}}=[p/\sqrt{b}+q_1\sqrt{(\pm 1)/\sqrt{b}}]^n$.

$\quad x/\sqrt{b}-y_1\sqrt{(\pm 1)/\sqrt{b}}=[p/\sqrt{b}-q_1\sqrt{(\pm 1)/\sqrt{b}}]^n$.

$$\therefore\ x=\frac{\sqrt{b}}{2}\{[p/\sqrt{b}+q_1\sqrt{(\pm 1)/\sqrt{b}}]^n+[p/\sqrt{b}-q_1\sqrt{(\pm 1)/\sqrt{b}}]^n\},$$

$$y=\frac{\sqrt{b}}{2\sqrt{\pm 1}}\{[p/\sqrt{b}+q_1\sqrt{(\pm 1)/\sqrt{b}}]^n-[p/\sqrt{b}-q_1\sqrt{(\pm 1)/\sqrt{b}}]^n\}.$$

By ascribing to n the values 1, 2, 3, ..., as many solutions as are desired can be obtained.

100. Proposed by A. H. BELL, Hillsboro, Ill.

Prove that every indeterminate equation of the second degree can be reduced to $x^2-Ay^2=Bz^2$. [*Legendre.*]

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Let $ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0$ represent any determinate equa-

tion of the second degree, in which x' and y' are the two indeterminates, and a, b, c, f, g, h being integers.

Solving this equation as a quadratic in x', and transposing $hy'+g$, we have

$$ax'+hy'+g=\pm\sqrt{[(h^2-ab)y'^2+2(hg-af)y'+(g^2-ac)]}....(1).$$

In order that the values of x' and y' may be positive integers, the expression in (1) under the radical, which we may denote by $Ay'^2+2qy'+r$, must be a perfect square; that is,

$$Ay'^2+2qy'+r=t^2, \text{ suppose, } ....(2).$$

Solving (2) as a quadratic in y', we have, after squaring each member,

$$(Ay'q+)^2=At^2+(q^2-Ar)....(3).$$

Now set $Ay'+q=u$, $q^2-Ar=B$, and transpose At^2, then (3) becomes

$$u^2-At^2=B....(4).$$

Since u and t may be fractional, assume $u=x/z$ and $t=y/z$, then substitute in (4), and we get $x^2-Ay^2=Bz^2$.

Also solved by the *PROPOSER*.

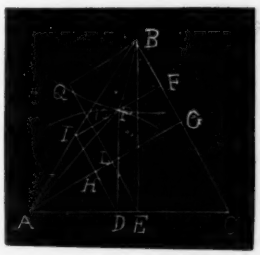
AVERAGE AND PROBABILITY.

120. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

If a given ellipse be placed at random on another equal ellipse, find the chance that the center of the first will fall on the second.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the one ellipse, center B, move parallel to itself so that it is always tangent to the other fixed ellipse, center A, at some point P. Let AC, BC be the directions of the axes-majores of the ellipses each semi-axes a, b. Let A be the origin, AC axis of abscissas; (x, y), coördinates of P; (m, n), coördinates of B; $\theta=\angle ACB$. Draw PF, AG perpendicular to BC; PD, BE perpendicular to AC; BQ parallel to AG; DI, EQ parallel to BC.



Then $AG=AL+BQ=m\sin\theta+n\cos\theta$,

$BG=QE-LE=n\sin\theta-m\cos\theta$.

$PF=AG-AH-IP=(m-x)\sin\theta+(n-y)\cos\theta$.

$BF=BG+HD-DI=(n-y)\sin\theta-(m-x)\cos\theta$.

∴ The equation to ellipse, center B, is

$$\frac{[(m-x)\cos\theta-(n-y)\sin\theta]^2}{a^2}+\frac{[(m-x)\sin\theta+(n-y)\cos\theta]^2}{b^2}=1....(1).$$

The equation to ellipse, center A, is

$$x^2/a^2+y^2/b^2=1....(2).$$

The invariant of these equations are

$$\triangle=\triangle_1=-a^4b^4.$$
$$\Theta=a^2b^2[b^2(m^2+n^2)+c(m\sin\theta+n\cos\theta)^2-c^2\sin^2\theta-3a^2b^2].$$
$$\Theta_1=a^2b^2[b^2m^2+a^2n^2-c^2\sin^2\theta-3a^2b^2), \text{ where } c=a^2-b^2.$$

Let $m=r\cos\varphi$, $n=r\sin\varphi$.

$$\therefore\ \Theta=a^2b^2\{r^2[a^2\sin^2(\theta+\varphi)+b^2\cos^2(\theta+\varphi)]-c^2\sin^2\theta-3a^2b^2\}$$
$$=a^2b^2(r^2A-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

$$\Theta_1=a^2b^2[r^2(a^2\sin^2\varphi+b^2\cos^2\varphi)-c^2\sin^2\theta-3a^2b^2]$$
$$=a^2b^2(r^2B-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

The condition that the ellipses should touch is

$$(\Theta\Theta_1-9\triangle\triangle_1)^2=4(\Theta^2-3\triangle\Theta_1)(\Theta_1^2-3\triangle_1\Theta).$$

The locus of B is

$$A^2B^2r^8-2(A+B)[(5a^2b^2+c^2\sin^2\theta)AB-2a^2b^2(A^2+B^2)]r^6$$
$$+[(A^2+B^2)(c^4\sin^4\theta-6a^2b^2c^2\sin^2\theta-27a^4b^4)+2AB(2c^4\sin^4\theta$$
$$+12a^2b^2c^2\sin^2\theta+27a^4b^4)]r^4-2c^4\sin^4\theta(A+B)(3a^2b^2+c^2\sin^2\theta)r^2$$
$$+c^6\sin^6\theta(4a^2b^2+c^2\sin^2\theta)=0.$$

This equation represents two loci, one when the moving ellipse is tangent externally, the other when the moving ellipse is tangent internally.

If a is the average area of the former, the chance is $p=\pi ab/a$.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of the pentagon formed by joining five random points on the surface of a given circle.

Solution by the PROPOSER.

Let a=radius of given circle, A=its area, $\triangle$=required average, $\triangle'$=average area when the five points are taken on both the circle A and a concentric annulus B, $\triangle_1$=average area when four points are taken on A and one on B.

Then $(\triangle'+\triangle)A=5B(\triangle_1-\triangle)$. But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=(A+B)\triangle/A$. $\therefore\ \triangle=\frac{5}{6}\triangle_1$.

Let one point O be on the circumference of the circle and the other four points P, Q, R, S anywhere on its surface. Let $OP=u$, $OS=x$, $OR=y$, $OQ=z$, $\angle AOE=\theta$, $\angle BOE=\varphi$, $\angle COE=\psi$, $\angle DOE=\delta$. Then $OA=2a\sin\theta=u'$, $OB=2a\sin\varphi=x'$, $OC=2a\sin\psi=y'$, $OD=2a\sin\delta=z'$.

Area $OPQRS=\frac{1}{2}uz\sin(\theta-\delta)+\frac{1}{2}yz\sin(\delta-\psi)+\frac{1}{2}xy\sin(\psi-\varphi)=C$.

$$\therefore\ \triangle=\frac{5}{6}\triangle_1=\frac{5\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz}{6\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}uxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz}$$

$$=\frac{20}{\pi^4a^8}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz$$

$$=\frac{20}{3\pi^4a^6}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}\int_0^{y'}[4au\sin(\theta-\delta)\sin\delta+4ay\sin(\delta-\psi)\sin\delta$$

$$+3xy\sin(\psi-\varphi)]uxy\sin^2\delta\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy$$

$$=\frac{160}{9\pi^4a^5}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}\int_0^{x'}[3u\sin(\theta-\delta)\sin\delta+3x\sin(\psi-\varphi)\sin\psi$$

$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]ux\sin^2\delta\sin^2\psi\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx$$

$$=\frac{320}{9\pi^4a}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}\int_0^{u'}[3u\sin(\theta-\delta)\sin\delta+4a\sin(\psi-\varphi)\sin\psi\sin\varphi$$

$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]u\sin^2\delta\sin^2\psi\sin^2\varphi\,d\theta\,d\delta\,d\psi\,d\varphi\,du$$

$$=\frac{2560a^2}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}\int_0^{\psi}[\sin(\theta-\delta)\sin\theta\sin\delta+\sin(\delta-\psi)\sin\delta\sin\psi$$

$$+\sin(\psi-\varphi)\sin\psi\sin\varphi]\sin^2\theta\sin^2\delta\sin^2\psi\sin^2\varphi\,d\theta\,d\delta\,d\psi\,d\varphi$$

$$=\frac{320a^2}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_0^{\delta}[4\sin(\theta-\delta)\sin\theta\sin\delta(\psi-\sin\psi\cos\psi)+3\sin^2\psi$$

$$+4\sin(\delta-\psi)\sin\delta\sin\psi(\psi-\sin\psi\cos\psi)-\sin^4\psi-3\psi\sin\psi\cos\psi]\sin^2\theta\sin^2\delta\sin^2\psi\,d\theta\,d\delta\,d\psi$$

$$=\frac{10a^2}{27\pi^4}\int_0^{\pi}\int_0^{\theta}[105\delta-72\delta^2\sin\delta\cos\delta+84\delta\sin^2\delta-120\delta\sin^4\delta-105\sin\delta\cos\delta$$

$$-82\sin^3\delta\cos\delta+32\sin^5\delta\cos\delta+96\sin(\theta-\delta)\sin\theta\sin\delta(\delta-\sin\delta\cos\delta)^2]\sin^2\theta\sin^2\delta d\theta d\delta$$

$$=\frac{5a^2}{162\pi^4}\int_0^{\pi}(360\theta^2-144\theta^3\sin\theta\cos\theta-360\theta^2\sin^4\theta+252\theta^2\sin^2\theta+336\theta\sin^5\theta\cos\theta$$

$$-132\theta\sin^3\cos\theta-615\theta\sin\theta\cos\theta+72\sin^8\theta-250\sin^6\theta-301\sin^4\theta+255\sin^2\theta)\sin^2\theta d\theta$$

$$=\frac{5a^2}{12\pi}\left(5-\frac{1001}{144\pi^2}\right).$$

MISCELLANEOUS.

116. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Prove that

$$-\left|\alpha^2\beta^2\gamma^2\right|^2\begin{vmatrix}a&b&c\\b&d&e\\c&e&f\end{vmatrix}^2=\begin{vmatrix}\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\\\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\\\begin{vmatrix}a&b&c&\gamma_1\\b&d&e&\gamma_2\\c&e&f&\gamma_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\gamma_1\\b&d&e&\gamma_2\\c&e&f&\gamma_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\gamma_1\\a&b&c&\gamma_2\\c&e&f&\gamma_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\end{vmatrix}$$

[From Muir's *Determinants.*]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, y, z be the minors with respect to a_1, a_2, a_3 in the first row, u, v, w the minors with respect to β_1, β_2, β_3 in the second row, and r, s, t the minors with respect to γ_1, γ_2, γ_3 in the third row of the determinant $\triangle$ on the right hand side of the equality. Then

$$\triangle=-\begin{vmatrix}a_1x-a_2y+a_3z,&\beta_1x-\beta_2y+\beta_3z,&\gamma_1x-\gamma_2y+\gamma_3z\\a_1u-a_2v+a_3w,&\beta_1u-\beta_2v+\beta_3w,&\gamma_1u-\gamma_2v+\gamma_3w\\a_1r-a_2s+a_3t,&\beta_1r-\beta_2s+\beta_3t,&\gamma_1r-\gamma_2s+\gamma_3t\end{vmatrix}$$

$$=\begin{vmatrix}a_1&a_2&a_3\\\beta_2&\beta_2&\beta_3\\\gamma_1&\gamma_2&\gamma_3\end{vmatrix}\begin{vmatrix}x&y&z\\u&v&w\\r&r&t\end{vmatrix}=(a_1\beta_2\gamma_3)\begin{vmatrix}x&y&z\\u&v&w\\r&s&t\end{vmatrix}$$

$$=(a_1\beta_2\gamma_3)\begin{vmatrix} \begin{vmatrix} b & c & a_1 \\ d & e & a_2 \\ e & f & a_3 \end{vmatrix} & \begin{vmatrix} a & c & a_1 \\ b & e & a_2 \\ c & f & a_3 \end{vmatrix} & \begin{vmatrix} a & b & a_1 \\ b & d & a_2 \\ c & e & a_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \beta_1 \\ d & e & \beta_2 \\ e & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & c & \beta_1 \\ b & e & \beta_2 \\ c & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & b & \beta_1 \\ b & d & \beta_2 \\ c & e & \beta_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \gamma_1 \\ d & e & \gamma_2 \\ e & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & c & \gamma_1 \\ b & e & \gamma_2 \\ c & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & b & \gamma_1 \\ b & d & \gamma_2 \\ c & e & \gamma_3 \end{vmatrix} \end{vmatrix}$$

Let A, B, C, etc., be the minors with respect to a, b, c, etc. Then

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2 = \begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}$$

$$\therefore \Delta=(a_1\beta_2\gamma_3)\begin{vmatrix} a_1A-a_2B+a_3C, & a_1B-a_2D+a_3E, & a_1C-a_2E+a_3F \\ \beta_1A-\beta_2B+\beta_3C, & \beta_1B-\beta_2D+\beta_3E, & \beta_1C-\beta_2E+\beta_3F \\ \gamma_1A-\gamma_2B+\gamma_3C, & \gamma_1B-\gamma_2D+\gamma_3E, & \gamma_1C-\gamma_2E+\gamma_3F \end{vmatrix}$$

$$=-(a_1\beta_2\gamma_3)^2\begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}=-(a_1\beta_2\gamma_3)^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2.$$

117. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x\cos\alpha+y\cos\alpha=a\cos\theta+b\cos\varphi$, and $x\sin\alpha+b\sin\varphi=y\sin\alpha+a\sin\theta=\kappa$, find the maximum value of κ, and the values of x and y.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$(x+y)\cos\alpha=a\cos\theta+b\cos\varphi....(1).$$
$$(x+y)\sin\alpha+a\sin\theta+b\sin\varphi=2\kappa....(2).$$

(1) in (2) gives

$$\frac{(a\cos\theta+b\cos\varphi)\sin\alpha}{\cos\alpha}+a\sin\theta+b\sin\varphi=2\kappa, \text{ or}$$

$$a\sin(\theta+\alpha)+b\sin(\varphi+\alpha)-2\kappa\cos\alpha=0=u.$$

$\therefore du/d\theta=a\cos(\theta+\alpha)=0$, $du/d\varphi=b\cos(\varphi+\alpha)=0$.

$\therefore \theta=\varphi=\frac{1}{2}\pi-\alpha$ for a maximum.

$\therefore \kappa=\frac{1}{2}(a+b)\sec\alpha$ is the maximum value.

$$x\cos\alpha+\frac{(\kappa-a\sin\theta)\cos\alpha}{\sin\alpha}=a\cos\theta+b\cos\varphi.$$

$$\therefore x = \frac{a\sin(\theta+\alpha)+b\cos\varphi\sin\alpha-\kappa\cos\alpha}{\sin\alpha\cos\alpha}.$$

$$y=\cos\alpha+\frac{(\kappa-b\sin\varphi)\cos\alpha}{\sin\alpha}=a\cos\theta+b\cos\varphi.$$

$$\therefore y = \frac{b\sin(\varphi+\alpha)+a\cos\theta\sin\alpha-\kappa\cos\alpha}{\sin\alpha\cos\alpha}.$$

When $\theta=\varphi=\frac{1}{2}\pi-\alpha$ and $\kappa=\frac{1}{2}(a+b)\sec\alpha$,

$$x=\frac{a-b\cos2\alpha}{\sin2\alpha},\quad y=\frac{b-a\cos2\alpha}{\sin2\alpha}.$$

118. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Show how to determine the illumination at any point of the surface of the water at the bottom of a deep well, due to the light from the sky.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let I=illumination, dQ the quantity of light that falls upon a small area dA of an illuminated surface, coming from an element of any bright surface dS.

Let O be the center of the element of the luminous surface, C the center of the illuminated area dA, and let $OC=r$, depth of well$=a$. Let θ=inclination of OC to normal at O, φ=inclination of OC to normal at C. Now if dA subtend solid angle $d\omega$ at O, and μ be the intrinsic brightness of dS, then

$$dQ=\mu dS\cos\theta d\omega, \text{ but } d\omega=\frac{dA\cos\varphi}{r^2}.$$

$\therefore dQ=\mu dS dA\cos\theta\cos\varphi/r^2$.

$\therefore I=dQ/dA=\mu\cos\varphi\cos\theta dS/r^2$.

If $d\sigma$ be the solid angle at C subtended by dS, then $d\sigma=\cos\theta dS/r^2$.

$\therefore I=\mu\cos\varphi d\sigma$.

The solid angle at C cannot be larger than the top of the well will admit. Let b=the area of the top of well, then we may write $I=\mu b/a^2$ for center of surface of water, since $\theta=\varphi=0$, and $S/r^2=b/a^2$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

170. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Find by strictly quadratic methods at least one set of values of x and y in the equations $x^2y^2+x=38$ and $xy+y^2=15$.

171. Proposed by IDA M. SCHOTTENFELTZ, A. M.

$ay^2+a=bxy+cx$, $bx^2+b=axy+cy$. Solve for x and y.

172. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Without solving the algebraically solvable quintic $y^5+py^3+\frac{1}{5}p^2y+r=0$, prove that it is irreducible in the domain of rationality (p, r).

GEOMETRY.

193. Proposed by PROFESSOR BEYENS.

Si le rapport du segment d'une base de la sphère à l'hemisphère est m/n, le rapport de l'hauteur du segment à deux bases qui resultera au rayon est égal à $2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$. [Problem 9699, *Educational Times.*]

194. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Glass paper weights, having the form of a regular tetrahedron, are to be packed for shipment, each in a paper box. Wanted to know the size and shape of the smallest box for the purpose. How much empty space in each box?

CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly toward the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

MECHANICS.

150. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

O is a point in the plane of a triangle, ABC, and D, E, F are the mid-points of the sides. Show, geometrically, that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

DIOPHANTINE ANALYSIS.

109. Proposed by HARRY S. VANDIVER, Bala, Pa.

If $m+n+1$ is a prime integer, show that $m!\times n! -(-1)^{\frac{1}{2}(3m-n)}$ is divisible by $m+n+1$. For instance, $6!\times 4!-(-1)^7$ is divisible by 11.

110. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Prove the results stated in the foot-note at the end of §4 in the article by L. E. Dickson in the October number of the MONTHLY.

111. Proposed by HARRY S. VANDIVER, Bala, Pa.

Show that: If a rational, integral polynomial of the nth degree in x, becomes a prime for more than n values of x, then it cannot be resolved into rational factors. (Test with the expression $x^4+2x^3+x^2+2x+1$.)

AVERAGE AND PROBABILITY.

135. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

If the line joining two points taken at random in the surface of a given circle be the diagonal of a square, the chance that the square lies wholly within the circle is $2-4/\pi$.

NOTES.

Prof. Lon C. Walker is doing postgraduate work in Leland Stanford Jr. University. F.

Prof. Harry S. Vandiver is taking a course of mathematics in the University of Pennsylvania. F

On December 15, the University of Klausenberg will celebrate the hundredth anniversary of the birth of Johann Bolyai. F.

Dr. F. Schottky of Marburg has been named to succeed the late L. Fuchs as Professor in the University of Berlin. Dr. K. Hensel of Berlin has been named Professor in the University of Marburg. D.

Dr. Joseph Swain, President of the University of Indiana since 1893, and formerly Professor of Mathematics in Indiana and Stanford Universities, was installed as President of Swarthmore College on November 15. D.

In the reconstitution of the University of London, complete courses of study in the various faculties of the University College have been established.

Mention may be made of the endowment of the department of pure mathematics by Mr Astor. The University is to be no longer merely an examining board.

D.

The address, November 13, of Dr. E. W. Hobson, retiring president of the London Mathematical Society, will be on the "The infinite and the infinitesimal in mathematical analysis." At this meeting the triennial De Morgan medal is to be presented to A. G. Greenhill for his investigations in pure and applied mathematics.

D.

The centenary of the birth of Abel, at Christiania, September 4-7, was regarded as a national event in Norway. The King of Norway and Sweden and his son Prince Eugen made a special journey from Stockholm. Among the mathematicians making addresses were H. Weber, Volterra, L. Sylow, Forsyth, Picard, Schwarz, Zeuthen, Henzel, and Mittag-Leffler. By a special act of the Storthing, power had been granted to the University of Christiania to confer honorary degrees on this special occasion. Among the twenty-nine mathematicians elected *Doctores Mathematicae* were Lord Kelvin, Lord Rayleigh, Dr. Salmon, Sir George Stokes, Prof. G. A. Darwin, Prof. A. R. Forsyth, Prof. Simon Newcomb, and Prof. J. Willard Gibbs.

D.

Since the last issue of the MONTHLY two more of its valued friends and contributors have received their summons to close their earthly toil. Prof. P. H. Philbrick and Hon. Josiah H. Drummond will no more give us personal encouragement in our work, nor gladden the hearts of those who loved to study their contributions. While they are dead, yet they will long live in the memories of many of the readers of the MONTHLY. We give below a brief sketch of the life of Prof. Philbrick and in the next issue will appear a brief biography of Hon. Josiah H. Drummond, whose death occurred on Saturday, October, 25.

Philetus Harry Philbrick was born at Machias, New York, March 8, 1839 and died at Bedford, Oregon, October 10, 1902. His death was not expected and was due to heart failure in the early stages of a mild typhoid infection.

After the age of ten years, he had but a few weeks schooling, until at twenty, when he sent himself to the Tafton Collegiate Seminary at Tafton, (now Bloomington) Wisconsin. Here he attended school until 1862, when he enlisted in Company A, 20th Wisconsin Regiment and served until the end of the war. After returning from the war, he taught school for a few months at Prairie du Chein, Wisconsin, and then entered the University of Michigan 1866. He graduated from that institution in the class of 1868, having in two years completed courses entitling him to the degrees of C. E. and B. Sc. For five years until 1873, he was engaged in various constructions in Michigan. He then received the appointment as professor of civil engineering in the University of Iowa and held this position until 1887 when he resigned to enter upon active field work in southern Louisiana, with headquarters at Lake Charles. This remained his home until within a few weeks of his death. Because of advancing years and chronic malarial disease, he removed to southern Oregon where he anticipated great en-

joyment in his remaining years looking after a fruit farm, solving problems in various departments of mathematics and writing articles on mathematcial, scientific, and other subjects.

During his residence in Louisiana, he was chief engineer of the Kansas, Watkins & Gulf Railroad; Lake Charles & Gulf Railroad; North American Land and Timber Co. etc.

Professor Philbrick was a contributor to various mathematical and scientific journals, including *The Analyst*, *The Mathematical Magazine*, *The Mathematical Visitor*, *The Engineering News*, and *The Mathematical Monthly*.

In each of the mathematical journals mentioned above, appear a great many of his excellent solutions of problems of various kinds. In the *Mathematical Magazine* is a brief article of his in which he developes a new method for finding the superior limes of a root of an algebraic equation. His method has considerable merit, and is, in general, superior to the common method. In the *Mathematical Magazine* also appears an article of his "On the Abuse of Logarithms." This elicited a reply by Prof. Herbert A. Howe of the University of Denver, Denver, Colorado, and occasioned a long friendly controversey.

In 1901 Professor Philbrick wrote a Field Manual for Engineers. This work which was published by John Wiley & Sons, will take its place among the best works on that subject for many years to come. It is well written and has received the highest commendation from prominent civil engineers.

Professor Philbrick was a man of keen intellect. He was entirely frank and fearless in the expression of his views. He wrote many controversial articles championing the Scientific Education, Metric System, Spelling Reform, Liberal Religion, etc. Three grown children mourn the loss of the kindest of fathers. F.

BOOKS AND PERIODICALS.

The Foundations of Geometry. By David Hilbert. Authorized Translation by E. J. Townsend, Ph, D., University of Illinois. Chicago: The Open Court Publishing Co. 1902. Pages, vii+132.

Readers of the American Mathematical Monthly, may consult a technical review of this translation in *Science*, Vol. XVI. No. 399. Aug. 22, 1902. pp. 307-8, where in the interest of merest justice are pointed out some few among the blemishes in what Professor Townsend puts forth as a translation of Hilbert's beautiful 'Festschrift.' These blemishes are the more indefensible because Professor Townsend had before him, in addition to the limped original, the admirable French translation of L. Langel.

For example, Hilbert, so studiously sparing of words, uses the word *Erklarung* nine times on his first thirteen pages.

Townsend never renders it at all. Where he adds from Langel, he seems to have no better luck with his French than with his German. For example, p. 25, "This axiom gives us nothing directly concerning the existence of limiting points or of the idea of converg-

ence" is how he renders, 'Cet axiome ne nous dit rien sur l' existence de points limites ni sur la notion de convergence.'

On p. 125, the translation reads: "We easily see that the criterion of theorem 44 is fulfilled, and, consequently, it follows that every regular polygon can be constructed by the drawing of straight lines and the laying off of segments." From this we should suppose that Professor Townsend studied his geometry from the popular treatise of Mr. Wentworth between 1877 and 1887, which during those years contained on p. 224, proposition XIII § 387:

"To inscribe a regular polygon of any number of sides in a given circle."

GEORGE BRUCE HALSTED.

The Book We Need. By Leon Stefflre, LL. B., of Bowdle, S. Dak. 8vo. Cloth. 218 pages. Price $1.00. San Francisco: The Whitaker & Ray Co.

The title of this little book does not in the least suggest the nature of the subject matter. There are many books that are urgently needed by most of us and perhaps this is one that is needed by some of us. The book is really an elementary arithmetic in which are introduced a few deviations from customary usage. In the first place, the author uses the Greek letter delta, inverted, instead of the decimal point.

The work closes with an epilogue in which the author advances a number of novel ideas.

The printing is not very good as there is too much uniformity in type throughout.

B. F. F.

The Business Man's Arithmetic. By Prof. J. S. Hunter. 8vo. Stiff Paper Back. 71 pages. Price 25cts. San Francisco: The Whitaker & Ray Co.

In this little pamphlet, the author has attempted to develop a system of computation, simple, brief, and sufficiently comprehensive as to be applicable to any kind of problem in any kind of business, and which presents no elaborate rules to be memorized by the learner. The author has simply made cancellation the basis of his system. This principle, I presume, is used already by nine-tenths of the practical computers of the country. Even this book in the hands of one unacquainted with fundamental principles would accomplish very little. A book to be put in the hands of pupils should always present principles and not rules or "systems." B. F. F.

Elementary Arithmetic of the Octimal Notation. By Geo. H. Cooper. 8vo. Stiff Paper Back. 70 pages. Price, 25cts. San Francisco: The Whitaker & Ray Co.

The object of this little work seems to be to revolutionize the present decimal system of notation. The author says, "Any attempt to perpetuate the use of the decimal system is nothing short of a crime against humanity, since it fails in every department." It has long been observed that some other than the decimal system would be more convenient, for example, the duo-decimal system in which 12 is the radix. But to change a system which has grown up with the race, and which answers so admirably the purposes which it is to serve, as does the decimal system of notation, seems to me exceedingly futile. Would it not be far better to spend our energies in advocating a universal adoption of the decimal system of weights and measures, rather than waste our powers in trying to establish an entirely new system? B. F. F.

Differential and Integral Calculus. By Virgil Snyder, Ph. D. and John I. Hutchinson, Ph. D., of Cornell University. 8vo. Cloth. xvi+320 pages. Price, $2.50. New York and Chicago: The American Book Co.

This book compares very favorably with the others of the Cornell Mathematical Series. The part of the work on Difierential Calculus is based largely upon McMahon & Snyder's Differential Calculus while the part on the Integral Calculus is entirely new.

The exercises are new and are carefully graded. Numerous illustrative examples are worked out and accompanied by helpful suggestions. The Derivative is presented vigorously as a limit. The treatment throughout the book is simple, clear, practical, and thoroughly rigorous, and in the hands of a live instructor will accomplish great good.

B. F. F.

The School Vistior An Elemenary Monthly Journal, Devoted to Difficult Work in Common School Studies. Price, $1.00 per year in advance. Published bp John S. Royer & Sons, Columbus, O.

The School Visitor was started in 1880, by Professor Royer, and its publication continued for 15 years, at the end of which time, owing to the nervous strain it caused and the tax it levied on the vital force of its editor, it was discontinued until 1900.

It is now appearing with all the vigor of its earlier days, and we trust that it may long continue to cultivate the minds and gladden the hearts of thousands of teachers who avail themselves of its influence.

The School Visitor is the most practical and stimulating periodical that the ordinary teacher can read. During the first year of its publication, it received contributions in mathematics from Prof. E. B. Seitz, Dr. Artemas Martin, Prof. Henry Gunder, and Dr. William Hoover, and the contributions of these gentlemen were sources of inspiration at that time to the writer. The *Visitor* in its mathematical department has at present the support of our valued contributor, Dr. G. B. M. Zerr, who solves every difficult problem that appears in it.

B. F. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published quarterly, under the auspices of Johns Hopkins University. Price, $5.00 per year.

The October number contains the following articles: On systems of Linear Differential Equations of the First Order, by Maxime Bocher; On the Quarternary Linear Homogenious Group and the Ternary Fractional Group, by T. M. Putnam; On Cardinal Numbers by A. N. Whitehead; On a Method of Constructing all the Groups of Order p^m, by G. A. Miller; Non-Euclidean Properties of Plane Cubics and of their First and Second Polars, by Henry Freeman Stecker.

B. F. F.

Annals of Mathematics. Published quarterly, under the auspices of Harvard University. Price, $2.00 per year in advance..

The October number cantains the following articles: The Geodesic Lines on an Anchor Ring, by Dr. G. A. Bliss; Proof of a Theorem concerning Isosceles Triangles, by Prof. H. F. Blichfeldt; An Elementary Exposition of Frobenius's Theory of Group-Characters and Group-Determinants, by Dr. L. E. Dickson; Communication concerning Mr. Ransom's Mechanical Construction of Conics, by Dr. E. V. Huntington.

B. F. F.

Periodicals Received: *The American Monthly Review of Reviews; The Literary Digest; Popular Astronomy*, November Number; *Monthly Weather Review; Scientific American; Mathematical Gazette; The Mathematical Messenger; The University Herald; The Ohio Educational Monthly; The Onio Teacher; Le Matematiche; Periodico di Matematica; School Science; The School Visitor; The Open Court.*

ERRATA.

Page 247, middle, in η_6' the term $+\rho_2\eta_5$ is omitted.

In solution of problem 112, Miscellaneous, "$A-C+B-D$" should read $A-C-B+D$, and the corresponding corrections throughout.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. DECEMBER, 1902. No. 12.

CLOSED LOXODROMICS OF THE TORUS.

By PROFESSOR ARNOLD EMCH, University of Colorado.

1. In Vol. VI (1899), pp. 136-139,* of this journal, I have proved the following theorem:†

If the ratio R/r of a torus is a rational fraction, and if a loxodromic of the torus winds m times around the axis (z-axis) and n times around the axial circle ($x^2+y^2=R^2$), then every loxodromic intersecting the first orthogonally closes also and their numbers m_1 and n_1 are related to the corresponding numbers m and n of the first by the equation

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{R^2-r^2}{r^2},$$

where R is the radius of the axial circle and r that of a meridian-section of the torus, and $R>r$.

If the equatorial plane (plane of the axial circle) be assumed as the xy-plane, u as the angle which the plane of a meridian (circle with center C and

* Slight corrections appear on p. 188, where a superfluous = occurs. ED. D.

† A torus, anchor-ring, is a surface of revolution generated by a circle which rotates about a fixed axis of its plane. A loxodromic on the torus is a curve which intersects all parallels (and consequently all meridians) at constant angles.

A clear discussion of the torus and its conformal representation may be found in F. Klein's little book: "Ueber Riemann's Theorie der Algebraischen Funktionen und ihrer Integrale." Leipzig, 1882, pp. 50-55.

radius r) makes with xz-plane, and v as the angle which the radius PC from C to a point P of this meridian makes with the perpendicular to the xy-plane, the rectangular coördinates of P may be expressed by

$$x=(R+r\sin v)\cos u,$$
$$y=(R+r\sin v)\sin u,$$
$$z=r\cos v,$$

and the (u, v)-equation of a closed loxodromic of the prescribed kind by

$$\sin\frac{n}{m}u=\frac{r+R\sin v}{R+r\sin v}\ldots.(1).$$

In this note I shall discuss the nature of these curves.

2. For this purpose substitute in (1) for $\sin u$ and $\sin v$ their equivalent expressions

$$\sin u=\frac{y}{\sqrt{(x^2+y^2)}},\quad \sin v=\frac{\sqrt{(x^2+y^2)}-R}{r}.$$

In this manner the Cartesian equation of the projection of the closed loxodromic on the (x, y)-plane is obtained. As n and m are positive integers, $\sin\frac{n}{m}u$ may be expressed algebraically in terms of $\sin u$. Hence, (1) becomes an algebraic expression in x and y, and we have the theorem:

The projection of a closed loxodromic of the torus on the xy-plane is an algebraic curve.

As the torus is a surface of the fourth order it follows further:

The loxodromic itself is algebraic.

3. I shall consider in particular the case where the loxodromic turns only once about the z-axis, *i. e.*, where $m=1$. Now

$$\sin nu=a_1\sin u+a_3\sin^3 u+\ldots.+a_n\sin^n u,\ \ldots.(2)$$

when $n=2k+1$ is odd; and

$$\sin nu=\cos u(b_1\sin u+b_3\sin^3 u+\ldots.+b_n\sin^{n-1}u),\ \ldots.(3)$$

when $n=2k$ is even.†

In the first case, $n=2k+1$, we obtain after setting $\sin u=\frac{y}{\sqrt{(x^2+y^2)}}$,

$$\sin(2k+1)u=\frac{y}{\sqrt{(x^2+y^2)}}\left[a_1+a_3\frac{y^2}{x^2+y^2}+\ldots. a_n\frac{y^{2k}}{(x^2+y^2)^k}\right],$$

* A. Emch: Ueber orthogonale Systeme und einige technische Anvendungen, Biel, 1898.

† See Jordan: Cours d'Analyse, Vol. 1, p. 238.